

Optimal Dividend Payments for the Piecewise-Deterministic Poisson Risk Model

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Abstract

This paper deals with optimal dividend payment problem in the general setup of a piecewise-deterministic compound Poisson risk model. The objective of an insurance business under consideration is to maximize the expected discounted dividend payout up to the time of ruin. Both restricted and unrestricted payment schemes are considered. In the case of restricted payment scheme, the value function is shown to be a classical solution of the corresponding Hamilton-Jacobi-Bellman equation, which, in turn, leads to an optimal restricted dividend payment policy. When the claims are exponentially distributed, the value function and an optimal dividend payment policy of the threshold type are determined in closed forms under certain conditions. The case of unrestricted payment scheme gives rise to a singular stochastic control problem. By solving the associated integro-differential quasi-variational inequality, the value function and an optimal barrier strategy are determined explicitly in exponential claim size distributions. Two examples are demonstrated and compared to illustrate the main results.

Key Words. Piecewise-deterministic compound Poisson model, optimal stochastic control, HJB equation, quasi-variational inequality, threshold strategy, barrier strategy.

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1 Introduction

The dividend problem in classical insurance risk models was originated in de Finetti [9], followed by revived interests in recent literature focusing on optimization of dividend payment

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strategies. The optimality is often considered to be a strategy which maximizes the expected present value of dividends received by the shareholders. Jeanblanc-Picqué and Shiryaev [16] and Asmussen and Taksar [2] investigated in diffusion models the dividend problems where the dividends are permitted to be paid out up to a maximal constant rate or a ceiling. We shall refer to such a type of dividend problem as restricted payment scheme. It was shown in their papers that the dividends should be paid out at the maximal admissible rate as soon as the surplus exceeds a certain threshold. Interestingly, it turns out that such a threshold strategy is the optimal restricted payment scheme in a variety of other risk models. For example, Gerber and Shiu [13] discussed the threshold strategy in the compound Poisson model and solved the problem explicitly when the claim size is exponentially distributed. Fang and Wu [10] studied a similar problem in the compound Poisson risk model with constant interest and showed the optimal dividend strategy is a threshold strategy for the case of an exponential claim distribution. See also Asmussen et al. [1], Bai and Paulsen [4], Choulli et al. [6], Paulsen and Gjessing [20], Schmidli [22], and references therein for some important developments on optimal dividend policies in the setting of controlled diffusions.

On the other hand, there has also been a significant amount of literature on dividend payment problems, where there is no such restriction of maximal rate imposed on dividend payment strategies. We shall refer to this type of dividend strategy as the unrestricted payment scheme. Unrestricted payment schemes are motivated by the fact that the boundedness of the dividend payment rate seems rather restrictive in many real-world applications. For instance, the insurance company is more likely to distribute the dividend once or twice a year; resulting unbounded payment rate. In such a scenario, the surplus level changes drastically on a dividend payday. In other words, the surplus level may displace abrupt or discontinuous changes due to “singular” dividend distribution policy. This gives rise to a singular stochastic control problem. Such problems are studied in Choulli et al. [6], Paulsen [18, 19], Paulsen and Gjessing [20], and the references therein when the surplus dynamics is modeled by a controlled diffusion. But to the best of our knowledge, related work in the setting of piecewise-deterministic compound Poisson risk model is relatively scarce. One exception is Schmidli [23, Section 2.4], which formulates and solves an optimal unrestricted payment problem when the surplus process follows a classical Cràmer-Lundberg risk model.

As pointed out in Cai et al. [5], the classical Cràmer-Lundberg risk model and the compound Poisson risk model with interest, absolute ruin are all special cases of piecewise-deterministic compound Poisson (PDCP) risk model. One naturally asks whether there exist unifying optimal solutions to both dividend payment schemes in piecewise-deterministic compound Poisson risk models. If so, can we confirm in general that the threshold strategy is the optimal restricted dividend policy whereas the barrier strategy is the optimal unrestricted dividend policy? We provide affirmative solutions to both questions in this paper under certain conditions.

The contribution and novelty of this work arise from several different aspects. First, we formulate and solve the problem within the framework of stochastic control theory in the

specific setting of piece-wise deterministic compound Poisson risk model. Roughly, the idea is to pay out the dividend at a dynamic rate in such a way that a certain reward function is optimized. Compared with the aforementioned related work in the setup of controlled diffusions, the associated Hamilton-Jacobi-Bellman equation in our work contains a non-local term (the integral term with respect to the claim size distribution), resulting substantial difficulty and technicality in the analysis. Nevertheless, we use renewal type arguments to overcome this difficulty and establish the Hamilton-Jacobi-Bellman equation. Furthermore, we obtain explicit solutions in exponential size distributions. Second, the generality of pure jump models in which both restricted and unrestricted payment schemes are presented and directly compared. Although special cases of the piecewise-deterministic compound Poisson risk model have been treated in the literature, this paper extends the spectrum of risk models which exhibit similar properties of optimality. Finally, it is worth mentioning that the solution methods presented in this paper can be more efficient alternatives of the approaches used in the existing literature.

The rest of the paper is organized as follows. The optimality of dividend strategies is formulated as a stochastic control problem in Section 2. In particular, we consider in Section 3 the restricted dividend payment schemes. We derive some properties of the value function and show that the value function is a classical solution to the Hamilton-Jacobi-Bellman equation (3.6). In Section 4, we formulate the optimal unrestricted payment scheme problem as a singular stochastic control problem. A verification theorem is established. Furthermore, under some fairly general conditions, we provide an explicit procedure to obtain an optimal dividend barrier and the corresponding optimal value function. When the claims are exponentially distributed, we obtain explicit solutions for both the restricted and unrestricted dividend payment schemes in Sections 3 and 4, respectively. Finally, the paper is concluded with several remarks in Section 5.

To facilitate later presentations, we introduce some notations here. We use I_A to denote the indicator function of a set A . Throughout the paper, we use the notations $\xi(t)$ and ξ_t interchangeably. A function ξ from $[0, \infty)$ to some Polish space E is *càdlàg* if it is right continuous and has left limits in E . When $E = \mathbb{R}$ and ξ is càdlàg, then $\Delta\xi(t) = \xi(t) - \xi(t-)$ for $t > 0$ and the convention $\Delta\xi(0) = \xi(0)$ is used. The continuous part of ξ is denoted by $\xi^c(t) := \xi(t) - \sum_{0 \leq s \leq t} \Delta\xi(s)$. As usual, $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$.

2 The Mathematical Model and Problem Formulation

To give a rigorous mathematical formulation of the optimization problem, we start with a filtered probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}\}$. We assume that in the absence of dividends, the surplus level is modeled by a piecewise-deterministic compound Poisson process.

Definition 2.1. A piecewise-deterministic compound Poisson (PDCP) process is a real-valued stochastic process $X = \{X(t), 0 \leq t < \infty\}$, defined on a given probability space

$\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}\}$. satisfying the following properties:

- (i) $X(0) = x \geq 0$,
- (ii) Let $0 = T_0 < T_1 < T_2 < \dots$ denote a sequence of jump points of the process X . Assume that $T_{i+1} - T_i$ has exponential distribution with mean $1/\lambda > 0$ for every $i = 0, 1, \dots$. Then the adapted counting process defined by $N(t) = \sum_{i=1}^{\infty} I_{\{T_i \leq t\}}$ follows a homogeneous Poisson process with intensity rate λ ,
- (iii) The jump sizes $Y_k = \Delta X(T_k) = X(T_k) - X(T_k-)$ for $k = 1, 2, \dots$ are independent and identically distributed nonnegative random variables with common distribution function $Q(y) = 1 - \bar{Q}(y) = \mathbb{P}\{Y_1 \leq y\}$, $0 \leq y < \infty$,
- (iv) The process between any two consecutive jumps is deterministic and given by

$$X_t = \phi_{X(T_k)}(t), \quad t \in [T_k, T_{k+1}), \quad k = 0, 1, 2, \dots,$$

where $\phi_z(t)$ is determined by

$$d\phi_z(t) = g(\phi_z(t)) dt, \quad t > 0,$$

satisfying $\phi_z(0) = z$ and $\lim_{t \rightarrow \infty} \phi_z(t) = L \in [-\infty, \infty]$. The function $g(x), x \in B$, satisfies the linear growth condition and is Lipschitz continuous on its domain B .

By virtue of [8] and [21], the generator of the PDCP is defined as

$$\mathcal{A}h(x) = g(x)h'(x) - \lambda h(x) + \lambda \int_0^{\infty} h(x-y) dQ(y), \quad x \in B, \quad (2.1)$$

where h is continuously differentiable.

As pointed out in Cai et al. [5], the class of PDCP processes includes many interesting risk models which appeared in the literature such as the compound Poisson risk models with interest, absolute ruin, dividend, and their respective dual models. The corresponding expressions for $g(x)$ and $\phi_x(t)$ for these specific models are as follows.

- In the classical compound Poisson model, the deterministic piece between any two consecutive claims is given by $g(x) = c, x \geq 0$. Hence, $\phi_x(t) = x + ct, x \geq 0$, and $L = \infty$.
- In the modification of the classical compound Poisson model where all positive surplus earns interest at rate $\rho > 0$, $g(x) = \rho x + c, x \geq 0$. Hence, $\phi_x(t) = (x + c/\rho)e^{\rho t}, x \geq 0$.

We now enrich the model by considering dividend payout. We denote by $D(t)$ the aggregate dividends by time t . We assume that $D = \{D(t), t \geq 0\}$ is càdlàg, nondecreasing, and

\mathcal{F}_t -adapted with $D(0-) = 0$. Moreover, we require that at any time t , the dividend payment should not exceed the current surplus level, i.e.,

$$\Delta D(t) = D(t) - D(t-) \leq X(t-).$$

Any dividend payment scheme $D = \{D(t), t \geq 0\}$ satisfying the above conditions is called an *admissible control* and the collection of all admissible controls is denoted by Π . The dynamics of the controlled surplus process under the admissible control D is

$$X^D(t) = X(t) - D(t) = x + \int_0^t g(X^D(s))ds - \sum_{i=1}^{N(t)} Y_i - D(t). \quad (2.2)$$

The ruin time is denoted by

$$\tau = \tau(x, D) := \inf \{t \geq 0 : X^D(t) < 0\}, \quad (2.3)$$

where $x \geq 0$ is the initial surplus.

The performance functional or the expected present value (EPV) of dividends up to ruin is defined as

$$J(x, \pi) = \mathbb{E}_x \int_0^\tau e^{-\delta t} dD(t), \quad (2.4)$$

where $\delta > 0$ is the force of interest. The objective is to find an admissible control $D^* \in \Pi$ that maximizes the performance functional. That is

$$V(x) := \sup_{D \in \Pi} \{J(x, D)\} = J(x, D^*). \quad (2.5)$$

Note that $V(x) = 0$ for all $x < 0$.

3 Restricted Payment Scheme

We first consider problem (2.5) for the case when the dividend payment scheme D is absolutely continuous with respect to time. That is, there exists some $u(t), t \geq 0$ such that

$$D(t) = \int_0^t u(s) ds.$$

Moreover, we assume that $u(t)$ is \mathcal{F}_t -adapted and that there exists some positive constant u_0 with $u_0 < g(x)$ for all $x \geq 0$ such that

$$0 \leq u(t) \leq u_0, \quad \forall t \geq 0.$$

Denote the collection of all such dividend payment schemes by Π_R . The EPV corresponding to the initial surplus $x \geq 0$ under the dividend payment policy $D = \{D(t), t \geq 0\}$ is given by

$$J(x, D) = \mathbb{E}_x \int_0^\tau e^{-\delta t} dD(t) = \mathbb{E}_x \int_0^\tau e^{-\delta t} u(t) dt. \quad (3.1)$$

The goal is to find an admissible policy $D_R^* \in \Pi_R$ such that

$$V_R(x) := \sup_{D \in \Pi_R} J(x, D) = J(x, D_R^*). \quad (3.2)$$

Note that a number of papers, including [10] and [13], consider the problem of seeking optimal restricted payment scheme in the classical compound Poisson model with or without interest.

Apparently, we have $V_R(x) \leq V(x)$ for all $x \geq 0$, where $V(x)$ is the value function defined in (2.5).

3.1 The Value Function

We first derive some elementary properties of the value function (3.2), which will help us to establish the HJB equation in Theorem 3.2.

Lemma 3.1. *The function $V_R(x)$ is bounded by u_0/δ , increasing, and Lipschitz continuous on $[0, \infty)$, and therefore absolutely continuous, and converges to u_0/δ as $x \rightarrow \infty$.*

Proof. That the function $V_R(x)$ is bounded by u_0/δ is obvious. The rest of the proof is divided into several steps.

Step 1. (Monotonicity) Let $x_2 > x_1 \geq 0$. Denote by $X_1(t)$ the surplus process under the dividend payment scheme $D_1 = \{\int_0^t u_1(s) ds, t \geq 0\}$ and initial surplus level x_1 . Put $\tau_1 := \inf \{t \geq 0 : X_1(t) < 0\}$. Also, let $D_2 = \{\int_0^t u_2(s) ds, t \geq 0\}$ be such that $u_2(t) = u_1(t)$ for all $t \leq \tau_1$. Let $X_2(t)$ be the surplus process under the dividend payment scheme $u_2(\cdot)$ and initial surplus level x_2 and $\tau_2 := \inf \{t \geq 0 : X_2(t) < 0\}$. Then apparently we have $\tau_2 \geq \tau_1$. Consequently it follows that

$$V_R(x_2) \geq J(x_2, D_2) = \mathbb{E} \int_0^{\tau_2} e^{-\delta t} u_2(t) dt \geq \mathbb{E} \int_0^{\tau_1} e^{-\delta t} u_1(t) dt = J(x_1, D_1).$$

Finally, taking supremum over $u_1(\cdot) \in \Pi_R$, it follows that $V_R(x_2) \geq V_R(x_1)$, as desired.

Step 2. (Lipschitz Continuity) Let h be a small positive value and $\tilde{D} \in \Pi_R$ be an arbitrary strategy. Define another strategy $D \in \Pi_R$ so that

$$D(t) = \begin{cases} \tilde{D}(t - h), & \text{if } T_1 \wedge t > h; \\ 0, & \text{otherwise.} \end{cases}$$

Denote by $X(t)$ the surplus process with initial surplus $x > 0$ under the dividend payment strategy D . It is clear that if $T_1 > h$, then the surplus at time h is $X(h) = \phi_x(h)$. Therefore it follows that

$$V_R(x) \geq J(x, D) \geq e^{-(\lambda+\delta)h} J(\phi_x(h), \tilde{D}),$$

which implies by taking the supremum over all possible strategies \tilde{D} that

$$V_R(x) \geq e^{-(\lambda+\delta)h} V_R(\phi_x(h)) \geq e^{-(\lambda+\delta)h} V_R(x), \quad (3.3)$$

with the last inequality from the fact that $V_R(x)$ is an increasing function and $g(x) \geq 0$. Thus $V_R(x)$ is right continuous by the continuity of $\phi_x(h)$ and the squeeze theorem. Letting $x = -\phi_x(h)$ in (3.3), we obtain

$$V_R(-\phi_x(h)) \geq e^{-(\lambda+\delta)h} V_R(x) \geq e^{-(\lambda+\delta)h} V_R(-\phi_x(h)). \quad (3.4)$$

Hence left continuity follows. Now it follows from (3.4) that

$$0 \leq V_R(x) - V_R(\phi_x(h)) \leq (1 - e^{-(\lambda+\delta)h}) V_R(x) \leq (1 - e^{-(\lambda+\delta)h}) u_0 / \delta.$$

Therefore, $V_R(x)$ is indeed Lipschitz continuous.

Step 3. (Limit at ∞) Let $D(t) := u_0 t$ for all $t \geq 0$. Denote the surplus process by $X(t)$ under the strategy D and initial surplus $x > 0$ and by τ the corresponding ruin time. Then as $x \rightarrow \infty$, τ converges to infinity. Therefore

$$V_R(x) \geq J(x, D) = \mathbb{E} \int_0^\tau e^{-\delta t} u_0 dt = \frac{u_0}{\delta} (1 - \mathbb{E}[e^{-\delta \tau}]) \rightarrow \frac{u_0}{\delta},$$

this, together with the boundedness of $V_R(x)$, leads to the desired conclusion. \square

3.2 The HJB Equation

We need the following *dynamic programming principle*. For any \mathcal{F}_t -stopping time θ ,

$$V_R(x) = \sup_{u(\cdot) \in \Pi_R} \mathbb{E}_x \left[\int_0^{\tau \wedge \theta} e^{-\delta s} u(s) ds + e^{-\delta(\theta \wedge \tau)} V_R(X(\theta \wedge \tau)) \right], \quad x \geq 0 \quad (3.5)$$

The dynamic programming principle is well-known in the diffusion case (see, for example, Fleming and Soner [11] and Krylov [17]). The dynamic programming principle is also proved for jump diffusions in Ishikawa [14]. In the case of PDMP, the dynamic programming principle can be found in Davis [8], see also Azcue and Muler [3]. The dynamic programming principle will help us to derive the Hamilton-Jacobi-Bellman (HJB) equation.

Theorem 3.2. *The function $V_R(x)$ is differentiable and fulfils the HJB equation*

$$\sup_{0 \leq u \leq u_0} \left\{ [g(x) - u] V_R'(x) - (\lambda + \delta) V_R(x) + \lambda \int_0^x V_R(x - y) dQ(y) + u \right\} = 0, \quad x \geq 0. \quad (3.6)$$

Proof. The proof is motivated by Schmidli [23, Theorem 2.32].

Step 1. Let $h > 0$ and $u \in [0, u_0]$. Let $\{D(t) := ut, t \geq 0\} \in \Pi_R$. Denote

$$\phi(t, x) = x + \int_0^t (g(\phi(s, x)) - u) ds, \quad t \geq 0.$$

The waiting time T_1 for the first claim has density $\lambda e^{-\lambda t}$ and T_1 is larger than h with probability $e^{-\lambda h}$. Using the law of total probability and taking $\theta = T_1 \wedge h$ in (3.5), we write

$$\begin{aligned} V_R(x) &\geq e^{-\lambda h} \left[\int_0^h e^{-\delta t} u dt + e^{-\delta h} V_R(\phi(h, x)) \right] \\ &\quad + \int_0^h \lambda e^{-\lambda t} \left[\int_0^t e^{-\delta s} u ds + e^{-\delta t} \int_0^{\phi(t, x)} V_R(\phi(t, x) - y) dQ(y) \right] dt. \end{aligned} \quad (3.7)$$

Rearranging the terms and dividing by h yields

$$\begin{aligned} &\frac{V_R(\phi(h, x)) - V_R(x)}{h} - \frac{1 - e^{-(\lambda+\delta)h}}{h} V_R(\phi(h, x)) + \frac{e^{-\lambda h}}{h} \int_0^h e^{-\delta t} u dt \\ &+ \frac{1}{h} \int_0^h \lambda e^{-\lambda t} \left[\int_0^t e^{-\delta s} u ds + e^{-\delta t} \int_0^{\phi(t, x)} V_R(\phi(t, x) - y) dQ(y) \right] dt \leq 0. \end{aligned} \quad (3.8)$$

Let

$$\begin{aligned} D^+ V_R(x) &= \limsup_{\Delta \rightarrow 0+} \frac{V_R(x + \Delta) - V_R(x)}{\Delta}, \\ D^- V_R(x) &= \liminf_{\Delta \rightarrow 0+} \frac{V_R(x + \Delta) - V_R(x)}{\Delta}. \end{aligned}$$

Note that $D^+ V_R(x)$ and $D^- V_R(x)$ are finite by Lipschitz continuity. Since $0 \leq u \leq u_0 < g(x)$, and noting the continuity of g , $\phi(\cdot, x)$ is strictly increasing and $\phi(t, x) \rightarrow x$ as $t \rightarrow 0$. Hence we can choose a sequence $\{h_n, n \geq 1\}$ satisfying $h_n \rightarrow 0$ as $n \rightarrow \infty$ and

$$\lim_{n \rightarrow \infty} \frac{V_R(\phi(h_n, x)) - V_R(x)}{\phi(h_n, x) - x} = D^+ V_R(x).$$

By the definition of $\phi(h_n, x)$, we have $\phi(h_n, x) \rightarrow x$ as $n \rightarrow \infty$. Also, we have from the continuity of g that as $n \rightarrow \infty$

$$\frac{\phi(h_n, x) - x}{h_n} = \frac{1}{h_n} \int_0^{h_n} [g(X(t)) - u] dt \rightarrow g(x) - u.$$

Hence it follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{V_R(\phi(h_n, x)) - V_R(x)}{h_n} &= \lim_{n \rightarrow \infty} \frac{V_R(\phi(h_n, x)) - V_R(x)}{\phi(h_n, x) - x} \cdot \frac{\phi(h_n, x) - x}{h_n} \\ &= D^+ V_R(x)(g(x) - u). \end{aligned} \quad (3.9)$$

Now taking $h = h_n$ in (3.8) and letting $n \rightarrow \infty$, in view of (3.9), detailed calculations reveal that

$$[g(x) - u] D^+ V_R(x) - (\lambda + \delta) V_R(x) + \lambda \int_0^x V_R(x - y) dQ(y) + u \leq 0, \quad \forall u \in [0, u_0]. \quad (3.10)$$

Step 2. On the other hand, by the definition of V_R in (3.2), there exists a strategy $\bar{D} := \{\int_0^t \bar{u}(s) ds, t \geq 0\} \in \Pi_R$ such that $J(x, \bar{D}) \geq V_R(x) - h^2$. Denote

$$\bar{\phi}(t, x) = x + \int_0^t (g(X(s)) - \bar{u}(s)) ds.$$

Then as argued before,

$$\begin{aligned} V_R(x) &\leq J(x, \bar{D}) + h^2 \\ &\leq h^2 + e^{-\lambda h} \left[\int_0^h e^{-\delta s} \bar{u}(s) ds + e^{-\delta h} V_R(\bar{\phi}(h, x)) \right] \\ &\quad + \int_0^h \lambda e^{-\lambda t} \left[\int_0^t e^{-\delta s} \bar{u}(s) ds + e^{-\delta t} \int_0^{\bar{\phi}(t, x)} V_R(\bar{\phi}(t, x) - y) dQ(y) \right] dt. \end{aligned}$$

We find by rearranging the terms and dividing by h that

$$\begin{aligned} h + \frac{V_R(\bar{\phi}(h, x)) - V_R(x)}{h} - \frac{1 - e^{-(\lambda+\delta)h}}{h} V_R(\bar{\phi}(h, x)) + \frac{e^{-\lambda h}}{h} \int_0^h e^{-\delta t} \bar{u}(t) dt \\ + \frac{1}{h} \int_0^h \lambda e^{-\lambda t} \left[\int_0^t e^{-\delta s} \bar{u}(s) ds + e^{-\delta t} \int_0^{\bar{\phi}(t, x)} V_R(\bar{\phi}(t, x) - y) dQ(y) \right] dt \geq 0. \end{aligned} \quad (3.11)$$

Denote

$$\bar{u} := \liminf_{s \rightarrow 0+} \bar{u}(s) \in [0, u_0].$$

Then it follows that

$$\limsup_{h \rightarrow 0+} \frac{\bar{\phi}(h, x) - x}{h} = \limsup_{h \rightarrow 0+} \frac{1}{h} \int_0^h [g(X(s)) - \bar{u}(s)] ds \leq g(x) - \bar{u}.$$

As in Step 1, we can choose a sequence $h_m \rightarrow 0+$ such that

$$\lim_{m \rightarrow \infty} \frac{V_R(\bar{\phi}(h_m, x)) - V_R(x)}{\bar{\phi}(h_m, x) - x} = D^- V_R(x).$$

Note that $D^- V_R(x) \geq 0$ by the monotonicity of V_R . Then it follows that

$$\begin{aligned} \limsup_{m \rightarrow \infty} \frac{V_R(\bar{\phi}(h_m, x)) - V_R(x)}{h_m} &= \limsup_{m \rightarrow \infty} \frac{V_R(\bar{\phi}(h_m, x)) - V_R(x)}{\bar{\phi}(h_m, x) - x} \cdot \frac{\bar{\phi}(h_m, x) - x}{h_m} \\ &\leq D^- V_R(x) [g(x) - \bar{u}]. \end{aligned}$$

Now by taking $h = h_m$ and letting $m \rightarrow \infty$ in (3.11), we obtain

$$[g(x) - \bar{u}] D^- V_R(x) - (\lambda + \delta) V_R(x) + \lambda \int_0^x V_R(x - y) dQ(y) + \bar{u} \geq 0. \quad (3.12)$$

Step 3. Note that $g(x) - \bar{u} \geq g(x) - u_0 > 0$. Hence, by taking $u = \bar{u}$ in (3.10) and comparing the resulting equation with (3.12), we have

$$D^+ V_R(x) \leq D^- V_R(x).$$

But $D^+V_R(x) \geq D^-V_R(x)$ by definition. Thus it follows that $D^+V_R(x) = D^-V_R(x)$ or $V_R(x)$ is differentiable from the right. Moreover, a combination of (3.10) and (3.12) yields that $V'_R(x+)$, the right derivative of V_R , satisfies the HJB equation

$$\sup_{0 \leq u \leq u_0} \left\{ [g(x) - u]V'_R(x+) - (\lambda + \delta)V_R(x) + \lambda \int_0^x V_R(x-y) dQ(y) + u \right\} = 0. \quad (3.13)$$

Similarly, we obtain that the left derivative $V'_R(x-)$ exists and fulfils the HJB equation

$$\sup_{0 \leq u \leq u_0} \left\{ [g(x) - u]V'_R(x-) - (\lambda + \delta)V_R(x) + \lambda \int_0^x V_R(x-y) dQ(y) + u \right\} = 0. \quad (3.14)$$

Step 4. With (3.13) in hands, we claim that

$$V'_R(x+) \geq 1 \Leftrightarrow (\lambda + \delta)V_R(x) - \lambda \int_0^x V_R(x-y) dQ(y) \geq g(x). \quad (3.15)$$

In fact, if $V'_R(x+) > 1$, then

$$\begin{aligned} 0 &= \sup_{0 \leq u \leq u_0} \left\{ [g(x) - u]V'_R(x+) - (\lambda + \delta)V_R(x) + \lambda \int_0^x V_R(x-y) dQ(y) + u \right\} \\ &= g(x)V'_R(x+) - (\lambda + \delta)V_R(x) + \lambda \int_0^x V_R(x-y) dQ(y). \end{aligned}$$

Hence we have

$$(\lambda + \delta)V_R(x) - \lambda \int_0^x V_R(x-y) dQ(y) = g(x)V'_R(x+) > g(x).$$

Conversely, if $(\lambda + \delta)V_R(x) - \lambda \int_0^x V_R(x-y) dQ(y) > g(x)$, then we have

$$\begin{aligned} 0 &= \sup_{0 \leq u \leq u_0} \left\{ [g(x) - u]V'_R(x+) - (\lambda + \delta)V_R(x) + \lambda \int_0^x V_R(x-y) dQ(y) + u \right\} \\ &< \sup_{0 \leq u \leq u_0} \{ (g(x) - u)(V'_R(x+) - 1) \}. \end{aligned}$$

But $g(x) - u \geq g(x) - u_0 > 0$. Thus we must have $V'_R(x+) > 1$. Hence the first case in (3.15) follows. Similar arguments establish the other two cases in (3.15).

Similarly, (3.14) leads to

$$V'_R(x-) \geq 1 \Leftrightarrow (\lambda + \delta)V_R(x) - \lambda \int_0^x V_R(x-y) dQ(y) \geq g(x). \quad (3.16)$$

Hence, (3.15) and (3.16) imply that $V'_R(x+)$ and $V'_R(x-)$ are both less than 1, both greater than 1, or both equal to 1. This, together with the HJB equations (3.13) and (3.14), implies that $V'_R(x+) = V'_R(x-)$ and so $V'_R(x)$ exists. Moreover, the continuities of V_R and g implies that $V'_R(x)$ is continuous. That is, $V_R(x)$ is continuously differentiable and satisfies the HJB equation (3.6). \square

3.3 An Optimal Strategy

Note that the HJB equation (3.6) is linear in u . The maximum value of the expression in the left hand side of (3.6) is achieved when $u = 0$ or $u = 1$, corresponding to whether $V'_R(x) > 1$ or $V'_R(x) < 1$, respectively. If $V'_R(x) = 1$, then u can be any value in $[0, 1]$. In view of this observation, we propose an optimal strategy $D_R^* = \{\int_0^t u_R^*(s) ds, s \geq 0\}$ with

$$u_R^*(t) = \begin{cases} 0, & \text{if } V'_R(X_R^*(t)) > 1, \\ u_0, & \text{if } V'_R(X_R^*(t)) \leq 1, \end{cases} \quad (3.17)$$

where $X_R^*(t)$ is the corresponding surplus process under the strategy (3.17).

Theorem 3.3. *The strategy (3.17) is optimal, i.e.,*

$$J(x, D_R^*) = V_R(x).$$

Proof. Let u_R^* and X_R^* as in above, and denote $\tau := \inf \{t \geq 0 : X_R^*(t) < 0\}$. Applying the Itô formula, we have

$$\begin{aligned} e^{-\delta(t \wedge \tau)} V_R(X_R^*(t \wedge \tau)) &= V_R(x) + \int_0^{t \wedge \tau} e^{-\delta s} [(g(X_R^*(s)) - u_R^*(s)) V'_R(X_R^*(s)) - \delta V_R(X_R^*(s))] ds \\ &\quad + \sum_{i=1}^{N_{t \wedge \tau}} e^{-\delta T_i} [V_R(X_R^*(T_i)) - V_R(X_R^*(T_i-))], \end{aligned} \quad (3.18)$$

Note that

$$\begin{aligned} &\mathbb{E} \sum_{i=1}^{N_{t \wedge \tau}} e^{-\delta T_i} [V_R(X_R^*(T_i)) - V_R(X_R^*(T_i-))] \\ &= \lambda \mathbb{E} \int_0^{t \wedge \tau} e^{-\delta s} \left(\int_0^{X_R^*(s-)} V_R(X_R^*(s) - y) dQ(y) - V_R(X_R^*(s)) \right) ds. \end{aligned} \quad (3.19)$$

Therefore, taking expectation in (3.18), we have

$$\begin{aligned} &\mathbb{E} [e^{-\delta(t \wedge \tau)} V_R(X_R^*(t \wedge \tau))] - V_R(x) \\ &= \mathbb{E} \int_0^{t \wedge \tau} e^{-\delta s} \left[(g(X_R^*(s)) - u_R^*(s)) V'_R(X_R^*(s)) - (\delta + \lambda) V_R(X_R^*(s)) \right. \\ &\quad \left. + \lambda \int_0^{X_R^*(s)} V_R(X_R^*(s) - y) dQ(y) \right] ds. \end{aligned} \quad (3.20)$$

Combining with the HJB equation (3.6), it follows that

$$V_R(x) = \mathbb{E} [e^{-\delta(t \wedge \tau)} V_R(X_R^*(t \wedge \tau))] + \mathbb{E} \int_0^{t \wedge \tau} e^{-\delta s} u_R^*(s) ds. \quad (3.21)$$

Moreover, since $V_R(y) = 0$ if $y < 0$, we have

$$\begin{aligned} e^{-\delta(t \wedge \tau)} V_R(X_R^*(t \wedge \tau)) &= e^{-\delta t} V_R(X_R^*(t)) I_{\{t < \tau\}} + e^{-\delta \tau} V_R(X_R^*(\tau)) I_{\{\tau < t\}} \\ &= e^{-\delta t} V_R(X_R^*(t)) I_{\{t < \tau\}}. \end{aligned}$$

Therefore, in view of Lemma 3.1, the bounded convergence theorem leads to

$$\lim_{t \rightarrow \infty} \mathbb{E}[e^{-\delta(t \wedge \tau)} V_R(X_R^*(t \wedge \tau))] = 0.$$

Finally, by letting $t \rightarrow \infty$ in (3.21), we obtain that

$$V_R(x) = \mathbb{E} \int_0^\tau e^{-\delta s} u_R^*(s) ds = J(x, D_R^*).$$

This completes the proof. \square

3.4 Exponential Claims

In order to obtain an explicit solution to the HJB equation (3.6) and an optimal dividend payment policy, we assume that the claims Y_1, Y_2, \dots are independently and exponentially distributed with density function

$$p(y) = \alpha e^{-\alpha y}, \quad y \geq 0,$$

where α is some positive constant. In addition, we assume

Hypothesis A The integro-differential equation

$$g(x)\varphi'(x) - (\lambda + \delta)\varphi(x) + \lambda \int_0^x \varphi(x-y)\alpha e^{-\alpha y} dy = 0, \quad x > 0, \quad (3.22)$$

has a strictly increasing solution $\psi_1(x)$ and the differential equation

$$[g(x) - u_0]\varphi''(x) - [\alpha g(x) - \alpha u_0 + g'(x) - (\lambda + \delta)]\varphi'(x) - \alpha \delta \varphi(x) = 0, \quad x > 0, \quad (3.23)$$

has a bounded concave solution $\psi_2(x)$.

Theorem 3.4. *Under hypothesis A, assume there exists a unique number $d > 0$ such that ψ_1 is concave on $(0, d)$, $\psi_2'(d) > 0$, and that*

$$\frac{\psi_1(d)}{\psi_1'(d)} - \frac{\psi_2(d)}{\psi_2'(d)} = \frac{u_0}{\delta}. \quad (3.24)$$

Then the value function $V_R(x)$ is given by

$$V_R(x) = \begin{cases} \frac{\psi_1(x)}{\psi_1'(d)}, & \text{if } 0 \leq x < d, \\ \frac{u_0}{\delta} + \frac{\psi_2(x)}{\psi_2'(d)}, & \text{if } x \geq d. \end{cases} \quad (3.25)$$

Moreover, the optimal dividend payment policy is the threshold strategy

$$u^*(t) = \begin{cases} 0, & \text{if } 0 \leq X^*(t) < d, \\ u_0, & \text{if } X^*(t) \geq d, \end{cases} \quad (3.26)$$

where X^ is the corresponding controlled surplus process.*

Proof. Denote the function defined by the right-hand side of (3.25) by $\Psi(x)$. Note that Ψ is continuously differentiable with $\Psi'(d) = 1$. Since both ψ_1 and ψ_2 are concave functions, we must have $\Psi'(x) > 1$ for $0 \leq x < d$ and $\Psi'(x) < 1$ for all $x > d$. Hence by virtue of Theorems 3.2 and 3.3, it only remains to show that Ψ satisfies the HJB equation (3.6).

It is clear by definition that

$$g(x)\Psi'(x) - (\lambda + \delta)\Psi(x) + \lambda \int_0^x \Psi(x-y)\alpha e^{-\alpha y} dy = 0, \quad 0 \leq x < d.$$

Therefore Ψ solves the HJB equation (3.6) if we can show that

$$[g(x) - u_0]\Psi'(x) - (\lambda + \delta)\Psi(x) + \lambda \int_0^x \Psi(x-y)\alpha e^{-\alpha y} dy + u_0 = 0, \quad x \geq d. \quad (3.27)$$

To this end, we define

$$h(x) = \alpha e^{-\alpha x} \int_d^x e^{\alpha y} \psi_2(y) dy, \quad x \geq d.$$

Then it is straightforward to verify that $\alpha\psi_2(x) = h'(x) + \alpha h(x)$. Denote the left-hand side of (3.23) by LHS. Then

$$\begin{aligned} \text{LHS} &= [g(x) - u_0]\psi_2''(x) - [\alpha g(x) - \alpha u_0 + g'(x) - (\lambda + \delta)]\psi_2'(x) - \alpha(\lambda + \delta)\psi_2(x) \\ &\quad + \lambda h'(x) + \lambda \alpha h(x) \\ &= [g(x) - u_0]\psi_2''(x) + g'(x)\psi_2'(x) - (\lambda + \delta)\psi_2'(x) + \lambda h'(x) \\ &\quad + \alpha\{[g(x) - u_0]\psi_2'(x) - (\lambda + \delta)\psi_2(x) + \lambda h(x)\}. \end{aligned}$$

Therefore by multiplying $e^{\alpha x}$ on both sides of (3.23), we obtain the equation

$$\begin{aligned} e^{\alpha x} \{ &[g(x) - u_0]\psi_2''(x) + g'(x)\psi_2'(x) - (\lambda + \delta)\psi_2'(x) + \lambda h'(x) \} \\ &+ \alpha e^{\alpha x} \{ [g(x) - u_0]\psi_2'(x) - (\lambda + \delta)\psi_2(x) + \lambda h(x) \} = 0, \end{aligned}$$

which, in turn, implies that

$$\frac{d}{dx} \{ e^{\alpha x} ([g(x) - u_0]\psi_2'(x) - (\lambda + \delta)\psi_2(x) + \lambda h(x)) \} = 0.$$

Recall that $h(d) = 0$. Hence, dividing both sides of the equation above by $\psi_2'(d)$ and integrating the resulting equation from d to x produces

$$\begin{aligned} e^{\alpha x} \left\{ [g(x) - u_0] \frac{\psi_2'(x)}{\psi_2'(d)} - (\lambda + \delta) \frac{\psi_2(x)}{\psi_2'(d)} + \alpha \lambda \int_d^x \frac{\psi_2(y)}{\psi_2'(d)} e^{-\alpha(x-y)} dy \right\} \\ = e^{\alpha d} \left\{ [g(d) - u_0] \frac{\psi_2'(d)}{\psi_2'(d)} - (\lambda + \delta) \frac{\psi_2(d)}{\psi_2'(d)} \right\}. \end{aligned} \quad (3.28)$$

It follows from (3.22) that

$$g(d) \frac{\psi_1'(d)}{\psi_1'(d)} - (\lambda + \delta) \frac{\psi_1(d)}{\psi_1'(d)} = -\lambda \alpha \int_0^d \frac{\psi_1(y)}{\psi_1'(d)} e^{-\alpha(d-y)} dy.$$

This, together with (3.24), implies that

$$\begin{aligned}
& e^{\alpha d} \left\{ [g(d) - u_0] \frac{\psi'_2(d)}{\psi'_2(d)} - (\lambda + \delta) \frac{\psi_2(d)}{\psi'_2(d)} \right\} \\
&= e^{\alpha d} \left\{ [g(d) - u_0] \frac{\psi'_1(d)}{\psi'_1(d)} - (\lambda + \delta) \left(\frac{\psi_1(d)}{\psi'_1(d)} - \frac{u_0}{\delta} \right) \right\} \\
&= \frac{\lambda}{\delta} u_0 e^{\alpha d} - \lambda \alpha \int_0^d \frac{\psi_1(y)}{\psi'_1(d)} e^{\alpha y} dy.
\end{aligned} \tag{3.29}$$

Now dividing both sides of (3.28) with the right-hand side replaced by the last line of (3.29) gives

$$\begin{aligned}
& [g(x) - u_0] \frac{\psi'_2(x)}{\psi'_2(d)} - (\lambda + \delta) \frac{\psi_2(x)}{\psi'_2(d)} + \alpha \lambda \int_d^x \frac{\psi_2(y)}{\psi'_2(d)} e^{-\alpha(x-y)} dy \\
& - \frac{\lambda}{\delta} u_0 e^{-\alpha(x-d)} + \lambda \alpha \int_0^d \frac{\psi_1(y)}{\psi'_1(d)} e^{-\alpha(x-y)} dy = 0.
\end{aligned}$$

We obtain the following after rearranging the terms

$$\begin{aligned}
& [g(x) - u_0] \frac{\psi'_2(x)}{\psi'_2(d)} - (\lambda + \delta) \left(\frac{\psi_2(x)}{\psi'_2(d)} + \frac{u_0}{\delta} \right) + \alpha \lambda \int_d^x \frac{\psi_2(y)}{\psi'_2(d)} e^{-\alpha(x-y)} dy \\
& + \frac{\lambda}{\delta} u_0 - \frac{\lambda}{\delta} u_0 e^{-\alpha(x-d)} + \lambda \alpha \int_0^d \frac{\psi_1(y)}{\psi'_1(d)} e^{-\alpha(x-y)} dy + u_0 = 0,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
& [g(x) - u_0] \frac{d}{dx} \left(\frac{\psi_2(x)}{\psi'_2(d)} + \frac{u_0}{\delta} \right) - (\lambda + \delta) \left(\frac{\psi_2(x)}{\psi'_2(d)} + \frac{u_0}{\delta} \right) \\
& + \alpha \lambda \int_d^x \left(\frac{\psi_2(y)}{\psi'_2(d)} + \frac{u_0}{\delta} \right) e^{-\alpha(x-y)} dy + \lambda \alpha \int_0^d \frac{\psi_1(y)}{\psi'_1(d)} e^{-\alpha(x-y)} dy + u_0 = 0, \quad x > d.
\end{aligned}$$

Hence, (3.27) is proved by the definition in (3.25). This finishes the proof of the theorem. \square

Remark 3.5. Assume Hypothesis A. If $\psi'_2(0) > 0$ and

$$[g(0) - u_0] \psi'_2(0) - (\lambda + \delta) \psi_2(0) + u_0 = 0,$$

then similar arguments as those in the proof of Theorem 3.4 imply that $\frac{\psi_2(x)}{\psi'_2(0)} + \frac{u_0}{\delta}$ solves the HJB equation (3.6). Moreover, thanks to Hypothesis A, $\frac{\psi'_2(x)}{\psi'_2(0)} < 1$. Thus the value function is

$$V_R(x) = \frac{\psi_2(x)}{\psi'_2(0)} + \frac{u_0}{\delta},$$

and the optimal restricted dividend payment scheme is to pay the maximal rate u_0 until ruin.

Example 3.6. To illustrate our results, let us consider the special case when $g(x) \equiv c > 0$ and the claim size distribution

$$Q(y) = 1 - e^{-\alpha y}, \quad y \geq 0.$$

We seek for a strictly increasing solution $\psi_1(x)$ to the integro-differential equation

$$c\varphi'(x) - (\lambda + \delta)\varphi(x) + \lambda \int_0^x \varphi(x-y)\alpha e^{-\alpha y} dy = 0, \quad x > 0, \quad (3.30)$$

and a bounded concave solution $\psi_2(x)$ to the differential equation

$$(c - u_0)\varphi''(x) - [\alpha c - \alpha u_0 - (\lambda + \delta)]\varphi'(x) - \alpha\delta\varphi(x) = 0, \quad x > 0.$$

As argued in [12], the unique (up to a constant multiple) strictly increasing solution to (3.30) is

$$\psi_1(x) = (r + \alpha)e^{rx} - (s + \alpha)e^{sx},$$

where $-\alpha < s < 0 < r$ are the roots of

$$c\xi^2 - (\lambda + \delta - \alpha c)\xi - \alpha\delta = 0. \quad (3.31)$$

Similarly, the differential equation (3.6) has a unique (up to a constant multiple) bounded concave solution

$$\psi_2(x) = -e^{tx},$$

where t is the negative root of

$$(c - u_0)\xi^2 - (\lambda + \delta - \alpha c + \alpha u_0)\xi - \alpha\delta = 0. \quad (3.32)$$

By virtue of condition (3.24), we obtain

$$\frac{(r + \alpha)e^{rd} - (s + \alpha)e^{sd}}{r(r + \alpha)e^{rd} - s(s + \alpha)e^{sd}} = \frac{1}{t} + \frac{u_0}{\delta}.$$

Solve the above equation for d

$$d = \frac{1}{r - s} \ln \left[\frac{(s + \alpha)(\delta t - \delta s - stu_0)}{(r + \alpha)(\delta t - \delta r - rtu_0)} \right] = \frac{1}{r - s} \ln \left[\frac{s(s - t)}{r(r - t)} \right], \quad (3.33)$$

which agrees with (9.15) of Gerber and Shiu [13]. However, our approach is considerably simpler than their method of optimizations.

Assume that $d > 0$, or equivalently,

$$\frac{(s + \alpha)(\delta t - \delta s - stu_0)}{(r + \alpha)(\delta t - \delta r - rtu_0)} > 1. \quad (3.34)$$

We claim that ψ_1 is concave on the interval $(0, d)$. In fact, it is straightforward to verify that the function $\psi_1'(x) = r(r + \alpha)e^{rx} - s(s + \alpha)e^{sx}$ is decreasing on $(-\infty, b)$ and increasing on $[b, \infty)$, where

$$b = \frac{1}{r - s} \ln \left(\frac{s^2(s + \alpha)}{r^2(r + \alpha)} \right) = \frac{1}{r - s} \ln \left(\frac{s[(\lambda + \delta)s + \alpha\delta]}{r[(\lambda + \delta)r + \alpha\delta]} \right). \quad (3.35)$$

Therefore the desired concavity will follow if we can show that $d \leq b$. Recall that $-\alpha < s < 0 < r$. Thus a comparison between (3.33) and (3.35) reveals that it suffices to prove

$$\frac{s(s - t)}{r(r - t)} - \frac{s[(\lambda + \delta)s + \alpha\delta]}{r[(\lambda + \delta)r + \alpha\delta]} < 0, \quad (3.36)$$

or, equivalently,

$$\frac{(s - t)}{(r - t)} - \frac{[(\lambda + \delta)s + \alpha\delta]}{[(\lambda + \delta)r + \alpha\delta]} = \frac{(s - r)[\alpha\delta + t(\lambda + \delta)]}{(r - t)[(\lambda + \delta)r + \alpha\delta]} > 0. \quad (3.37)$$

But since

$$t = \frac{\lambda + \delta - \alpha(c - u_0) - \sqrt{(\lambda + \delta - \alpha(c - u_0))^2 + 4(c - u_0)\alpha\delta}}{2(c - u_0)},$$

detailed calculations reveals that $\alpha\delta + t(\lambda + \delta) < 0$, which, in turn, leads to (3.37). Therefore it follows that $\psi_1'(x)$ is decreasing on $(0, d)$ and hence $\psi_1(x)$ is indeed concave on $(0, d)$.

Therefore, according to Theorem 3.4, if $d > 0$, then the value function is

$$V_R(x) = \begin{cases} \frac{(r+\alpha)e^{rx} - (s+\alpha)e^{sx}}{r(r+\alpha)e^{rd} - s(s+\alpha)e^{sd}}, & \text{if } 0 \leq x < d; \\ \frac{u_0}{\delta} + \frac{1}{t}e^{t(x-d)}, & \text{if } x \geq d, \end{cases}$$

and the optimal restricted dividend payment scheme is the threshold strategy given in (3.26).

4 Unrestricted Payment Scheme

In Section 3, we considered the case when the dividend payment rate is bounded. Consequently, the surplus level changes continuously in time t in response to the dividend payment policy. However, in many real situations, the boundedness of the dividend payment rate seems rather restrictive. For instance, the insurance company is more likely to distribute the dividend once or twice a year; resulting unbounded payment rate. In such a scenario, the surplus level changes drastically on a dividend payday. In other words, the surplus level may displace abrupt or discontinuous changes due to “singular” dividend distribution policy. This is generally termed as a singular stochastic control problem in the literature (see, for example, Flemming and Soner [11]). In light of these discussions, we consider the (singular) optimal dividend payment policy for the piecewise deterministic Poisson risk model introduced in Section 2. Thus, throughout this section, $D(t)$, the total amount of dividends paid out up to time t , is not necessarily absolutely continuous with respect to t .

The following proposition can be proved using exactly the same arguments as those used in [24]. It indicates that the value function V defined in (2.5) is nondecreasing.

Proposition 4.1. *For any $0 \leq y \leq x$, we have*

$$V(x) \geq (x - y) + V(y). \quad (4.1)$$

4.1 The Verification Theorem

The following verification theorem will help us to find the value function and an optimal dividend payment strategy.

Theorem 4.2. *Suppose there exists a continuously differentiable function $\varphi : \mathbb{R} \mapsto \mathbb{R}_+$ satisfying $\varphi(y) = 0$ for $y < 0$ and that it solves the following quasi-variational inequality:*

$$\max \{(\mathcal{A} - \delta)\varphi(x), 1 - \varphi'(x)\} = 0, \quad x > 0, \quad (4.2)$$

(a) *Then $\varphi(x) \geq V(x)$ for every $x \geq 0$.*

(b) *Define the continuation region*

$$\mathcal{C} = \{x \geq 0 : 1 - \varphi'(x) < 0\}.$$

Assume there exists a dividend payment scheme $\pi^ = \{D^*(t) : t \geq 0\} \in \Pi$ and corresponding process X^* satisfying (2.2) such that,*

$$X^*(t) \in \bar{\mathcal{C}} \text{ for Lebesgue almost all } 0 \leq t \leq \tau, \quad (4.3)$$

$$\int_0^t [\varphi'(X^*(s)) - 1] dD^{*c}(s) = 0, \text{ for any } t \leq \tau, \quad (4.4)$$

$$\lim_{N \rightarrow \infty} \mathbb{E}_x [e^{-r(\tau \wedge N \wedge \beta_N)} \varphi(X^*(\tau \wedge N \wedge \beta_N))] = 0, \quad (4.5)$$

and if $X^(s) \neq X^*(s-)$, then*

$$\varphi(X^*(s)) - \varphi(X^*(s-)) = -\Delta D^*(s), \quad (4.6)$$

where $\beta_N := \inf\{t \geq 0 : |X^(t)| \geq N\}$. Then $\varphi(x) = V(x)$ for every $x \geq 0$ and π^* is an optimal dividend payment strategy.*

Proof. (a) Fix some $x \geq 0$ and $\pi = \{D(t) : t \geq 0\} \in \Pi$ and let \hat{X} denote the corresponding solution to (2.2). Choose N sufficiently large so that $x < N$ and define $\beta_N := \inf\{t \geq 0 : |\hat{X}(t)| \geq N\}$. By virtue of [15, Theorem 2.32],

$$\beta_N \rightarrow \infty \quad \text{a.s. as } N \rightarrow \infty. \quad (4.7)$$

Write $\tau_N := N \wedge \beta_N \wedge \tau$. Then Itô's formula ([8]) leads to

$$\begin{aligned} & \mathbb{E}_x [e^{-\delta\tau_N} \varphi(\hat{X}(\tau_N))] - \varphi(x) \\ &= \mathbb{E}_x \int_0^{\tau_N} e^{-\delta s} (\mathcal{A} - \delta) \varphi(\hat{X}(s)) ds - \mathbb{E}_x \int_0^{\tau_N} e^{-\delta s} \varphi'(\hat{X}(s)) dD^c(s) \\ & \quad + \mathbb{E}_x \sum_{0 \leq s \leq \tau_N} e^{-\delta s} [\varphi(\hat{X}(s)) - \varphi(\hat{X}(s-))]. \end{aligned}$$

It follows from (4.2) that

$$\begin{aligned} & \mathbb{E}_x[e^{-\delta\tau_N}\varphi(\hat{X}(\tau_N))] - \varphi(x) \\ & \leq -\mathbb{E}_x \int_0^{\tau_N} e^{-\delta s} \varphi'(\hat{X}(s)) dD^c(s) + \mathbb{E}_x \sum_{0 \leq s \leq \tau_N} e^{-\delta s} \Delta\varphi(\hat{X}(s)), \end{aligned}$$

where $\Delta\varphi(\hat{X}(s)) = \varphi(\hat{X}(s)) - \varphi(\hat{X}(s-))$. Applying the mean value theorem to $\Delta\varphi(\hat{X}(s))$, we obtain

$$\Delta\varphi(\hat{X}(s)) = \varphi'(\xi(s))\Delta\hat{X}(s),$$

where $\xi(s) = \theta(s)\hat{X}(s) + (1 - \theta(s))\hat{X}(s-)$ for some $\theta(s) \in (0, 1)$. Note that

$$\Delta\hat{X}(s) = -\Delta D(s) - \sum_{i=1}^{\infty} Y_i I_{\{T_i=s\}}.$$

Thus it follows that

$$\begin{aligned} \varphi(x) & \geq \mathbb{E}_x[e^{-\delta\tau_N}\varphi(\hat{X}(\tau_N))] + \mathbb{E}_x \int_0^{\tau_N} e^{-\delta s} \varphi'(\hat{X}(s)) dD^c(s) \\ & \quad + \mathbb{E}_x \sum_{0 \leq s \leq \tau_N} e^{-\delta s} \varphi'(\xi(s))\Delta D(s) + \mathbb{E}_x \sum_{0 \leq s \leq \tau_N} e^{-\delta s} \varphi'(\xi(s)) \sum_{i=1}^{\infty} Y_i I_{\{T_i=s\}}. \end{aligned}$$

Using (4.2) again and noting that φ is nonnegative, it follows that

$$\varphi(x) \geq \mathbb{E}_x \int_0^{\tau_N} e^{-\delta s} dD^c(s) + \mathbb{E}_x \sum_{0 \leq s \leq \tau_N} e^{-\delta s} \Delta D(s) = \mathbb{E}_x \int_0^{\tau_N} e^{-\delta s} dD(s).$$

Now letting $N \rightarrow \infty$, it follows from (4.7) and the bounded convergence theorem that

$$\varphi(x) \geq \mathbb{E}_x \int_0^{\tau} e^{-\delta s} dD(s) = J(x, \pi).$$

Finally, taking supremum over all $\pi \in \Pi$, we obtain $\varphi(x) \geq V(x)$, as desired.

(b) Let $\pi^* = \{D^*(t), t \geq 0\} \in \Pi$ satisfy (4.3)–(4.6). Define β_N and τ_N as before with X^* replacing \hat{X} . As in part (a), we have from Itô's formula that

$$\begin{aligned} & \mathbb{E}_x[e^{-\delta\tau_N}\varphi(X^*(\tau_N))] - \varphi(x) \\ & = \mathbb{E}_x \int_0^{\tau_N} e^{-\delta s} (\mathcal{A} - \delta)\varphi(X^*(s)) ds - \mathbb{E}_x \int_0^{\tau_N} e^{-\delta s} \varphi'(X^*(s)) dD^{*c}(s) \\ & \quad + \mathbb{E}_x \sum_{0 \leq s \leq \tau_N} e^{-\delta s} [\varphi(X^*(s)) - \varphi(X^*(s-))]. \end{aligned}$$

By (4.3), $(\mathcal{A} - \delta)\varphi(X^*(s)) = 0$ for almost all $s \in [0, \tau]$. This, together with (4.4) and (4.6), implies that

$$\varphi(x) = \mathbb{E}_x[e^{-\delta\tau_N}\varphi(X^*(\tau_N))] + \mathbb{E}_x \int_0^{\tau_N} e^{-\delta s} dD^*(s).$$

Letting $N \rightarrow \infty$ and using (4.5) and (4.7), we obtain

$$\varphi(x) = \mathbb{E}_x \int_0^\tau e^{-\delta s} dD^*(s).$$

This shows that $\varphi(x) = J(x, \pi^*) = V(x)$ for every $x \geq 0$ and π^* is an optimal dividend payment scheme. \square

4.2 Exponential Claims

In order to obtain an explicit solution to the quasi-variational inequality (4.2) and an optimal dividend payment policy, as in Section 3.4, we again assume that the claims Y_1, Y_2, \dots are independently and exponentially distributed with mean $1/\alpha$ for some $\alpha > 0$.

In what follows, we construct an explicit solution to (4.2), and verify that the solution is the value function defined in (2.5). To this end, we suppose

Hypothesis B. The integral-differential equation

$$(\mathcal{A} - \delta)\varphi(x) = g(x)\varphi'(x) - (\lambda + \delta)\varphi(x) + \lambda \int_0^x \varphi(x-y)\alpha e^{-\alpha y} dy = 0, \quad x > 0, \quad (4.8)$$

has a continuously differentiable and strictly increasing solution $\psi(x)$. Moreover, $\psi'(x)$ achieves its minimum value at $b > 0$ and $\psi'(x)$ is nondecreasing on (b, ∞) .

Theorem 4.3. *Under Hypothesis B, the solution to (4.2) is*

$$\Phi(x) = \begin{cases} \frac{\psi(x)}{\psi'(b)}, & \text{if } x \leq b, \\ x - b + \frac{\psi(b)}{\psi'(b)}, & \text{if } x > b. \end{cases} \quad (4.9)$$

Proof. Note that $\Phi \in C^1$. Obviously, if $x \leq b$, thanks to Hypothesis B, $\Phi(x)$ satisfies (4.2). If $x > b$, $\Phi'(x) = 1$. Therefore it remains to show that

$$g(x)\Phi'(x) - (\lambda + \delta)\Phi(x) + \lambda \int_0^x \Phi(x-y)\alpha e^{-\alpha y} dy \leq 0, \quad x > b. \quad (4.10)$$

To this end, we claim that

$$g(x)\Phi''(x) + [\alpha g(x) + g'(x) - (\lambda + \delta)]\Phi'(x) - \alpha\delta\Phi(x) \leq 0, \quad x > b. \quad (4.11)$$

By virtue of Hypothesis B, $\psi'(x) > 0$ for $x > 0$ and $\psi''(x) \geq 0$ for $x > b$. Hence it follows that

$$\begin{aligned} & g(x)\Phi''(x) + [\alpha g(x) + g'(x) - (\lambda + \delta)]\Phi'(x) - \alpha\delta\Phi(x) \\ & \leq g(x) \cdot \frac{\psi''(x)}{\psi'(x)} + [\alpha g(x) + g'(x) - (\lambda + \delta)] \frac{\psi'(x)}{\psi'(x)} - \alpha\delta \left(x - b + \frac{\psi(b)}{\psi'(b)} \right). \end{aligned} \quad (4.12)$$

But $\psi'(x)$ is nondecreasing on (b, ∞) , hence it follows that

$$x - b = \int_b^x \frac{1}{\psi'(y)} \psi'(y) dy \geq \frac{1}{\psi'(x)} \int_b^x \psi'(y) dy = \frac{1}{\psi'(x)} (\psi(x) - \psi(b)). \quad (4.13)$$

Since ψ is a solution to (4.8), by applying the operator $(\frac{d}{dx} + \alpha)$ to (4.8), we see by straightforward calculations that

$$g(x)\psi''(x) + [\alpha g(x) + g'(x) - (\lambda + \delta)]\psi'(x) - \alpha\delta\psi(x) = 0. \quad (4.14)$$

A combination of (4.12)–(4.14) leads to

$$\begin{aligned} & g(x)\Phi''(x) + [\alpha g(x) + g'(x) - (\lambda + \delta)]\Phi'(x) - \alpha\delta\Phi(x) \\ & \leq \frac{1}{\psi'(x)} [g(x)\psi''(x) + [\alpha g(x) + g'(x) - (\lambda + \delta)]\psi'(x) - \alpha\delta\psi(x)] + \alpha\delta\psi(b) \left(\frac{1}{\psi'(x)} - \frac{1}{\psi'(b)} \right) \\ & = 0 + \alpha\delta\psi(b) \left(\frac{1}{\psi'(x)} - \frac{1}{\psi'(b)} \right) \leq 0, \end{aligned}$$

where in the above, we have used the fact that $\psi'(x)$ is nondecreasing on (b, ∞) . Equation (4.11) is therefore established.

Next we show that Φ satisfies (4.10). In fact, as in the proof of Theorem 3.4, if we define

$$h(x) := \int_0^x \Phi(x - y)Q(dy) = \int_0^x \Phi(x - y)\alpha e^{-\alpha y} dy = e^{-\alpha x} \int_0^x \Phi(y)\alpha e^{\alpha y} dy,$$

then $\alpha\Phi(x) = h'(x) + \alpha h(x)$. Therefore by rearranging the terms in (4.11), we obtain

$$\begin{aligned} 0 & \geq g(x)\Phi''(x) + [\alpha g(x) + g'(x) - (\lambda + \delta)]\Phi'(x) - \alpha(\lambda + \delta)\Phi(x) + \alpha\lambda\Phi(x) \\ & = g(x)\Phi''(x) + [\alpha g(x) + g'(x) - (\lambda + \delta)]\Phi'(x) - \alpha(\lambda + \delta)\Phi(x) + \lambda h'(x) + \lambda\alpha h(x) \\ & = g(x)\Phi''(x) + g'(x)\Phi'(x) - (\lambda + \delta)\Phi'(x) + \lambda h'(x) \\ & \quad + \alpha [g(x)\Phi'(x) - (\lambda + \delta)\Phi(x) + \lambda h(x)]. \end{aligned}$$

Hence it follows that for any $x > b$, we have

$$\begin{aligned} 0 & \geq e^{\alpha x} [g(x)\Phi''(x) + g'(x)\Phi'(x) - (\lambda + \delta)\Phi'(x) + \lambda h'(x)] \\ & \quad + \alpha e^{\alpha x} [g(x)\Phi'(x) - (\lambda + \delta)\Phi(x) + \lambda h(x)] \\ & = \frac{d}{dx} [e^{\alpha x} (g(x)\Phi'(x) - (\lambda + \delta)\Phi(x) + \lambda h(x))]. \end{aligned}$$

Note that

$$g(b)\Phi'(b) - (\lambda + \delta)\Phi(b) + \lambda h(b) = 0.$$

Hence we conclude that for any $x > b$,

$$e^{\alpha x} (g(x)\Phi'(x) - (\lambda + \delta)\Phi(x) + \lambda h(x)) \leq e^{\alpha b} (g(b)\Phi'(b) - (\lambda + \delta)\Phi(b) + \lambda h(b)) = 0,$$

which, in turn, leads to (4.10). The theorem is thus established. \square

Since $\Phi(x)$ defined in (4.9) solves the quasi-variational inequality (4.2), thanks to Theorem 4.2, we have $V(x) \leq \Phi(x)$ for any $x \geq 0$. The next theorem implies that we can find an admissible dividend payment policy D^* such that $J(x, D^*) = \Phi(x)$ for any $x \geq 0$. Therefore it follows that $\Phi(x) = V(x)$ and that D^* is the optimal policy.

Theorem 4.4. *Under Hypothesis A, the strategy given by continuous part*

$$dD^*(t) = \begin{cases} 0, & \text{if } X_t < b, \\ g(b) dt, & \text{if } X_t = b, \end{cases} \quad (4.15)$$

and singular part

$$\Delta D^*(t) = X_t - b, \quad \text{if } X_t > b,$$

with $D^*(0-) = 0$ is the optimal control that corresponds to $\Phi(x)$ given in (4.9).

Proof. It is easy to verify that the strategy D^* and the corresponding surplus process X^* satisfy all the conditions in Theorem 4.2(b). Hence Theorems 4.2 and 4.3 imply that $J(x, D^*) = \Psi(x)$ for all $x \geq 0$.

Thanks to the special structure of the piecewise deterministic Poisson risk model, we present an alternative proof. Suppose $0 \leq x < b$. Denote $\pi^* = \{D^*(t), t \geq 0\}$ and $W(x) = J(x, \pi^*)$ with D^* given in (4.15). Using the law of total probability at $t = T_1 \wedge h$ for a sufficiently small $h > 0$ such that $\phi_x(h) \leq b$ and applying the strong Markov property,

$$\begin{aligned} W(x) &= \mathbb{E}_x \left[\int_0^t e^{-\delta s} dD^*(s) \right] + \mathbb{E}_x \left[\int_t^\tau e^{-\delta s} dD^*(s) \right] = \mathbb{E}_{X_t} [e^{-\delta t} W(X_t)] \\ &= e^{-(\lambda+\delta)h} W(\phi_x(h)) + \int_0^h \lambda e^{-(\lambda+\delta)s} \int_0^{\phi_x(s)} W(\phi_x(s) - y) dQ(y) ds. \end{aligned}$$

Since $\phi_x(t)$ is uniquely determined by

$$d\phi_x(t) = g(\phi_x(t)) dt, \quad 0 < t < T_1,$$

with $\phi_x(0) = x$. Denote $d = \phi_x(h)$. We make a change of variables by letting $z = \phi_x(s)$ and thus $ds = (1/g(z)) dz$ and

$$W(x) = e^{-(\lambda+\delta) \int_x^d \frac{1}{g(z)} dz} W(d) + \int_x^d \lambda e^{-(\lambda+\delta) \int_x^z \frac{1}{g(y)} dy} \int_0^z W(z - y) dQ(y) \frac{1}{g(z)} dz,$$

which clearly shows that $W(\cdot)$ is absolutely continuous. We denote the two terms on the right-hand side by I_1 and I_2 respectively. Thus, by taking derivatives on both sides with respect to x , we have

$$\begin{aligned} W'(x) &= \frac{\lambda + \delta}{g(x)} I_1 - \frac{\lambda}{g(x)} \int_0^x W(x - y) dQ(y) + \frac{\lambda + \delta}{g(x)} I_2 \\ &= \frac{\lambda + \delta}{g(x)} W(x) - \frac{\lambda}{g(x)} \int_0^x W(x - y) dQ(y), \end{aligned}$$

which shows that $W(\cdot)$ must satisfy (4.8). It is easy to show in a manner similar to Step 1 of Lemma 3.1 that $W(\cdot)$ must be strictly increasing and hence $W(\cdot) = C\psi(\cdot)$ for some constant $C > 0$ to be determined.

Clearly, when $x > b$, according to (4.15) and applying strong Markov property, we obtain

$$W(x) = x - b + W(b). \quad (4.16)$$

When $x = b$, the surplus level X stays at b until the time of first claim. Conditioning on the time of first claim, we have

$$\begin{aligned} W(b) &= \int_0^\infty \lambda e^{-\lambda t} \int_0^t e^{-\delta s} g(s) ds dt + \int_0^\infty \lambda e^{-(\lambda+\delta)t} \int_0^b W(b-y) dQ(y) dt \\ &= \frac{g(b)}{\lambda+\delta} + \frac{\lambda}{\lambda+\delta} \int_0^b W(b-y) dQ(y). \end{aligned}$$

Therefore,

$$C\psi(b) = \frac{g(b)}{\lambda+\delta} + C \frac{\lambda}{\lambda+\delta} \int_0^b \psi(b-y) dQ(y),$$

which implies

$$C = \frac{g(b)}{(\lambda+\delta)\psi(b) - \lambda \int_0^b \psi(b-y) dQ(y)} = \frac{1}{\psi'(b)}.$$

Therefore, together with (4.16), we complete the proof that $W(x) = \Phi(x)$ given in (4.9). \square

Example 4.5. Similar to Example 3.6, we consider a controlled piece-wise deterministic compound Poisson surplus process. As in Example 3.6, we take $g(x) = c > 0$ and $Q(y) = 1 - e^{-\alpha y}$, $y \geq 0$. But in contrast to Example 3.6, here we allow the optimal dividend payment policy to be singular.

Recall that $\psi(x) = (r+\alpha)e^{rx} - (s+\alpha)e^{sx}$ solves the integral-differential equation (4.8) and that $\psi'(x) = r(r+\alpha)e^{rx} - s(s+\alpha)e^{sx}$ achieves its unique minimum value at

$$b = \frac{1}{r-s} \ln \left(\frac{s^2(s+\alpha)}{r^2(r+\alpha)} \right) = \frac{1}{r-s} \ln \left(\frac{s[(\lambda+\delta)s+\alpha\delta]}{r[(\lambda+\delta)r+\alpha\delta]} \right)$$

and that ψ' is nondecreasing on (b, ∞) . Therefore in view of Theorems 4.3 and 4.4, if $b > 0$, then the dividend payment strategy defined in (4.15) is optimal and the value function is

$$V(x) = \begin{cases} \frac{(r+\alpha)e^{rx} - (s+\alpha)e^{sx}}{r(r+\alpha)e^{rb} - s(s+\alpha)e^{sb}} & \text{if } x < b, \\ x - b + \frac{(r+\alpha)e^{rb} - (s+\alpha)e^{sb}}{r(r+\alpha)e^{rb} - s(s+\alpha)e^{sb}} & \text{if } x \geq b. \end{cases}$$

On the other hand, if $b \leq 0$, then one can verify that the function $x \mapsto x + \frac{c}{\lambda+\delta}$ solves the quasi-variational inequality (4.2). Hence Theorem 4.2 implies that $W(x) \leq x + \frac{c}{\lambda+\delta}$. Moreover, the expected present value from the strategy of paying all surplus immediately is equal to $x + \frac{c}{\lambda+\delta}$. Hence it follows that

$$V(x) = x + \frac{c}{\lambda+\delta}, \quad \text{if } b \leq 0.$$

Using the fact that r and s are the roots of (3.31), we can verify that $b > 0$ if and only if

$$\alpha\lambda c > (\lambda + \delta)^2. \quad (4.17)$$

Hence we can summarize the value function as

$$V(x) = \begin{cases} \frac{(r+\alpha)e^{rx} - (s+\alpha)e^{sx}}{r(r+\alpha)e^{rb} - s(s+\alpha)e^{sb}} & \text{if } \alpha\lambda c > (\lambda + \delta)^2 \text{ and } x < b, \\ x - b + \frac{(r+\alpha)e^{rb} - (s+\alpha)e^{sb}}{r(r+\alpha)e^{rb} - s(s+\alpha)e^{sb}} & \text{if } \alpha\lambda c > (\lambda + \delta)^2 \text{ and } x \geq b, \\ x + \frac{c}{\lambda + \delta} & \text{if } \alpha\lambda c \leq (\lambda + \delta)^2. \end{cases}$$

Note that our result agrees that of Schmidli [23, p. 94]. But our approach is much simpler than theirs.

Finally we demonstrate the comparison of restricted and unrestricted payment schemes through a numerical example, in which $\alpha = 1$, $\delta = 0.1$, $c = 4$, $\lambda = 2$, and $u_0 = 3$. Note that both (3.34) and (4.17) are satisfied. The resulting unrestricted and restricted value functions $V(x)$ and $V_R(x)$ are shown in Figure 1 (a). We also plot the difference $V(x) - V_R(x)$ in Figure 1 (b). Note that the plot of $V_R(x)$ in Figure 1 (a) demonstrates the limit result of $V_R(x)$ presented in Lemma 3.1.

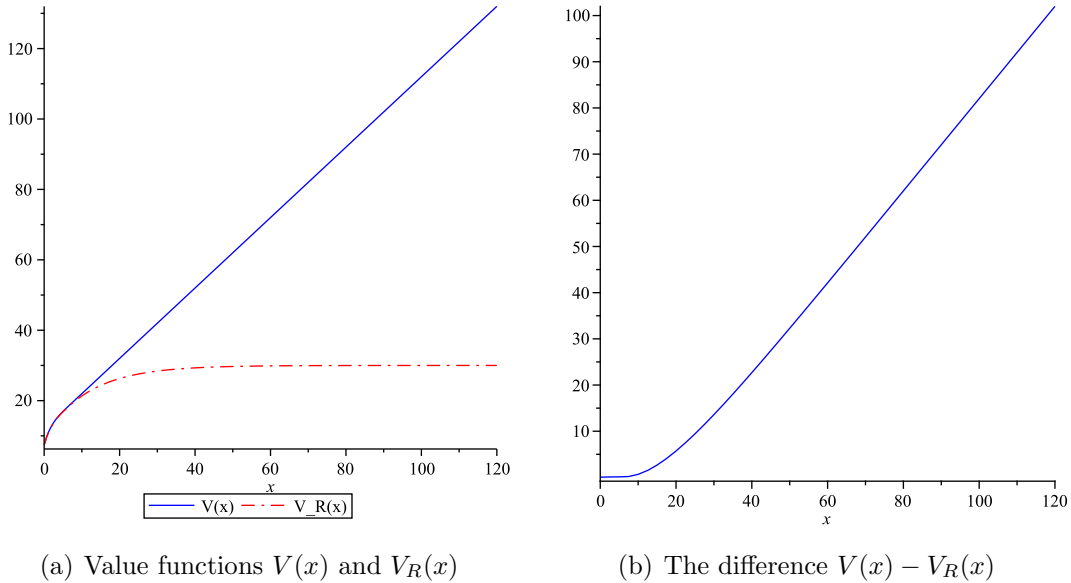


Figure 1: Comparison of the unrestricted and restricted value functions

5 Conclusions and Remarks

This work is devoted to the optimal dividend payment problem for the piecewise-deterministic compound Poisson risk model. Under certain conditions, it is shown that the optimal restricted dividend payment scheme is the threshold strategy, in which dividends are paid only

at the maximal rate when the surplus attains a certain level; and the optimal unrestricted dividend payment scheme is the barrier strategy, in which dividends are only paid at times when the surplus reaches a (possibly different) level and at such a rate that the surplus stays at the same level until claim arrival and/or ruin. To demonstrate the main results, two examples are calculated and compared in some details.

A number of questions deserve further investigations. In particular, one can consider more realistic models in which the parameters and hence the dynamics of the surplus level depend on a stochastic process such as a continuous-time Markov chain. In such a case, we need to deal with regime-switching jump diffusions ([25]) and the resulting HJB equation will be a coupled system of nonlinear integro-differential equations. It is conceivable that it will be much more challenging to obtain the corresponding value function and optimal dividend policy in closed forms. Some initial work in this vein can be found in [26]. It will also be interesting to approach the optimal dividend payment problem in the setting of piecewise deterministic compound Poisson risk model using the powerful viscosity solution framework (Crandall et al. [7]). Another problem of great interest is to consider transaction costs, reinsurance, and/or investments. Similar work in the setting of controlled diffusions can be found in [4, 6] and others.

References

- [1] S. Asmussen, B. Højgaard, and M. Taksar. Optimal risk control and dividend distribution policies. Example of excess-of loss reinsurance for an insurance corporation. *Finance Stoch.*, 4(3):299–324, 2000.
- [2] S. Asmussen and M. Taksar. Controlled diffusion models for optimal dividend pay-out. *Insurance Math. Econom.*, 20:1–15, 1997.
- [3] P. Azcue and N. Muler. Optimal reinsurance and dividend distribution policies in the Cramér-Lundberg model. *Math. Finance*, 15(2):261–308, 2005.
- [4] L. Bai and J. Paulsen. Optimal dividend policies with transaction costs for a class of diffusion processes. *SIAM J. Control Optim.*, 48(8):4987–5008, 2010.
- [5] J. Cai, R. Feng, and G.E. Willmot. On the expectation of total discounted operating costs up to default and its applications. *Adv. in Appl. Probab.*, 41(2):495–522, 2009.
- [6] T. Choulli, M. Taksar, and X.Y. Zhou. A diffusion model for optimal dividend distribution for a company with constraints on risk control. *SIAM J. Control Optim.*, 41(6):1946–1979, 2003.
- [7] Michael G. Crandall, Hitoshi Ishii, and Pierre-Louis Lions. User’s guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):1–67, 1992.

- [8] M.H.A. Davis. *Markov Models and Optimization*, volume 49 of *Monographs on Statistics and Applied Probability*. Chapman and Hall/CRC, London, 1993.
- [9] B. de Finetti. Su un' impostazione alternativa della teoria collettiva del rischio. *Transactions of the XVth International Congress of Actuaries*, 2:433–443, 1957.
- [10] Y. Fang and R. Wu. Optimal dividend strategy in the compound Poisson model with constant interest. *Stochastic models*, 23(1):149–166, 2007.
- [11] W.H. Fleming and H.M. Soner. *Controlled Markov Processes and Viscosity Solutions*, volume 25 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, New York, NY, second edition, 2006.
- [12] H. U. Gerber and E. S. W. Shiu. On the time value of ruin, with discussion and a reply by the authors. *N. Am. Actuar. J.*, 2(1):48–78, 1998.
- [13] H.U. Gerber and E.S.W. Shiu. On optimal dividend strategies in the compound Poisson model. *N. Am. Actuar. J.*, 10(2):76–93, 2006.
- [14] Y. Ishikawa. Optimal control problem associated with jump processes. *Appl. Math. Optim.*, 50(1):21–65, 2004.
- [15] J. Jacod and A.N. Shiryaev. *Limit Theorems for Stochastic Processes*. Springer-Verlag, New York, NY, second edition, 2003.
- [16] M. Jeanblanc-Picqué and A.N. Shiryaev. Optimization of the flow of dividends. *Russ. Math. Surv.*, 50(2):257–277, 1995.
- [17] N.V. Krylov. *Controlled diffusion processes*, volume 14 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, New York-Berlin, 1980. Translated from the Russian by A. B. Aries.
- [18] J. Paulsen. Optimal dividend payments until ruin of diffusion processes when payments are subject to both fixed and proportional costs. *Adv. in Appl. Probab.*, 39(3):669–689, 2007.
- [19] J. Paulsen. Optimal dividend payments and reinvestments of diffusion processes with both fixed and proportional costs. *SIAM J. Control Optim.*, 47(5):2201–2226, 2008.
- [20] J. Paulsen and H. Gjessing. Optimal choice of dividend barriers for a risk process with stochastic return on investments. *Insurance Math. Econom.*, 20(3):215–223, 1997.
- [21] Tomasz Rolski, Hanspeter Schmidli, Volker Schmidt, and Jozef Teugels. *Stochastic Processes for Insurance and Finance*. Wiley Series in Probability and Statistics. John Wiley & Sons, Ltd., Chichester, 1999.

- [22] H. Schmidli. On minimizing the ruin probability by investment and reinsurance. *Ann. Appl. Probab.*, 12(3):890–907, 2002.
- [23] H. Schmidli. *Stochastic Control in Insurance*. Springer-Verlag, London, 2008.
- [24] Q.S. Song, R. Stockbridge, and C. Zhu. On optimal harvesting problems in random environments. *SIAM J. Control Optim.*, 49(2):859–889, 2011.
- [25] G. Yin and C. Zhu. *Hybrid Switching Diffusions: Properties and Applications*, volume 63 of *Stochastic Modelling and Applied Probability*. Springer, New York, 2010.
- [26] C. Zhu. Optimal control of risk process in a regime switching environment. *Automatica*, 2010. To appear, [arxiv.org/1009.3247v3](https://arxiv.org/abs/1009.3247v3).